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DISINTEGRATION OF MEASURES ON COMPACT TRANSFORMATION GROUPS

BY

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ABSTRACT. Let G be a compact metrizable group which acts freely on a locally compact Hausdorff space X. Let μ be a measure on X, $\pi: X \to X/G \equiv Y$ the projection, $\nu = \pi(\mu)$. We show that there is a ν-Lusin-measurable disintegration of μ with respect to π. We use this result to prove a structure theorem concerning T-ergodic measures on bitransformation groups (G, X, T) with G metric and X compact. We finish with some remarks concerning the case when G is not metric.

Introduction. This paper falls naturally into two parts. The first deals with the following situation: G, a compact metric group, acts *freely* on a locally compact space X (thus, if $g \cdot x = x$ for any $x \in X$ and $g \in G$, then $g =$ identity in G). The quotient $Y = X/G$ is locally compact; let $\pi: X \to Y$ be the canonical projection. We show that each measure μ on X has a $\pi(\mu) = \nu$-Lusin-measurable disintegration with respect to π (see §0 for definitions; see [6] for a detailed discussion of disintegrations and their relationship to liftings). No theorem known to the author yields this result, although it is similar to theorems on the disintegration of a measure on a product space (see [2] and [6]).

The second part considers a special case: μ is a T-ergodic measure on a compact Hausdorff space X which is the phase space of a bitransformation group (G, X, T) with G metric. Let

$$G_0 = \left\{ g \in G \,\middle|\, \int_X f(gx)\, d\mu(x) = \int_X f(x)\, d\mu(x) \text{ for all } f \in C(X) \right\},$$

and let γ_0 be Haar measure on G_0. We show that, if $y \to \lambda_y$ is the disintegration of §1, then each λ_y "looks like" γ_0 in a certain sense. The following result is crucial: If Z is a Hausdorff space and $f: X \to Z$ a μ-Lusin-measurable, T-invariant map, then $f(x) = \text{const}$ μ-a.e. Finally, in §6, we remove the metrizability assumption on G; we assume the existence of a strong lifting on (Y, ν) (the only place in the paper where this is done).

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0. **Preliminaries.** We quote definitions and results from [1], [3], [6], and [12]; see these references for more details.

0.1. Let W be a locally compact Hausdorff space, $K(W)$ the set of continuous complex functions on W with compact support, with the topology of uniform convergence on compact sets. A (*Radon*) *measure* on W is a continuous complex linear functional on $K(W)$; we denote the set of all such functionals by $C^*(W)$. We will *always* assume $C^*(W)$ is given the topology of pointwise convergence (the *vague* topology). Let $M_+(W)$ consist of those *positive* elements η of $C^*(W)$ for which $\|\eta\| = \eta(W) < \infty$. See [1, Chapter III, §1; Chapter IV, §4, $\eta°7$, Proposition 12].

0.2. Let W be as above, η a positive measure on W, Z a topological space, π: $W \to Z$ a map. Say π is *η-Lusin-measurable* [12] if, for each compact $K \subset W$, there is a sequence (K_i) of pairwise disjoint compact sets such that (i) $\eta(K \sim \bigcup_{i=1}^{\infty} K_i) = 0$; (ii) $\pi | K_i$ is continuous ($i \geq 1$). If π is η-Lusin-measurable, then $\pi^{-1}(V)$ is η-measurable for each open (closed) $V \subset Z$. See [1, Chapter IV, §5].

0.3. Let W, η be as above. Let $M^1(W, \eta)$ be the set of η-integrable complex functions on W, $L^1(W, \eta)$ the (usual) set of equivalence classes modulo null sets. Let N_1 be the norm on $L^1(W, \eta)$. Similarly, define $M^{\infty}(W, \eta)$ and $L^{\infty}(W, \eta)$.

0.4. Let W, η, π be as above, and let Z be locally compact Hausdorff. Call π *η-proper* if, for each compact $K \subset Z$, $\pi^{-1}(K)$ is η-integrable. If π is η-proper, the map $f \to \eta(f \circ \pi)$: $K(Z) \to C$ defines a Radon measure γ on Z; we write $\gamma = \pi(\eta)$. If $f \in L^1(Z, \gamma)$, then $\int_W (f \circ \pi)\, d\mu = \int_Z f\, d\gamma$. See [1, Chapter V, §6].

0.5. Let λ: $W \to M_+(Z)$: $w \to \lambda_w$ be a map. Say λ is *weakly η-measurable* if $y \to \langle \lambda_w, f \rangle$ is η-measurable for each $f \in K(Z)$. The map λ is *weakly essentially η-integrable* if $f \in K(Z) \Rightarrow w \to \langle \lambda_w, f \rangle$ is essentially η-integrable. In this case, the formula $f \to \int_W \langle \lambda_w, f \rangle\, d\eta(w)$ defines a measure γ on Z. If $f \in L^1(Z, \gamma)$, then the map $w \to \langle \lambda_w, f \rangle$ is defined η-a.e., is η-integrable, and $\gamma(f) = \int_W \langle \lambda_w, f \rangle\, d\eta(w)$. We write $\gamma = \int_W \lambda_w\, d\eta(w)$. Finally, if λ': $W \to M_+(Z)$ is another map, say $\lambda' = \lambda$ *weakly ν-a.e.* if $\langle \lambda'_w, f \rangle = \langle \lambda_w, f \rangle$ ν-a.e. for each $f \in K(Z)$. See [1, Chapter V, §3].

0.6. DEFINITION. Let π: $W \to Z$ be η-proper, $\gamma = \pi(\eta)$. A weakly essentially γ-integrable map λ: $Z \to M_+(W)$: $z \to \lambda_z$ is a *disintegration of η with respect to π if*:

(a) λ is γ-adequate [1, Chapter V, §3, $\eta°1$, Definition 1];

(b) $\|\lambda_z\| = 1$ for all z;

(c) Support$(\lambda_z) \subset \pi^{-1}(z)$ $(z \in Z)$;
(d) $\eta = \int_Z \lambda_z \, d\gamma(z)$.
If λ is γ-Lusin-measurable (with respect to the vague topology) and satisfies $\|\lambda_z\| <$ const $< \infty$ γ-a.e., then λ is ν-adequate. See [1, Chapter VI, §3, η°1; Chapter V, §3, η°1, Proposition 2].

0.7. Theorem (Dunford and Pettis). *Let E be a separable Banach space, E' its dual with the norm topology. Let ζ: $L^1(\eta) \to E'$ be a bounded linear map. There exists a map λ: $W \to E'$: $w \to \lambda_w$ such that* (i) $w \to \langle e, \lambda_w \rangle$ *is η-measurable, and* (ii) *if $f \in L^1(\eta)$, then*

$$\langle \zeta(f), e\rangle = \int_W f(w)\langle \lambda_w, e\rangle \, d\eta(w) \qquad (e \in E).$$

One has $\|\zeta\| = \operatorname{ess\,sup}_{w \in W}\|\lambda_w\|$. *Further, if* λ': $W \to E'$ *is another map satisfying* (i) *and* (ii), *then $\lambda' = \lambda$ locally η - a.e.*

For a proof, see [6, Corollary 1, p. 89].

0.8. The following notation will be fixed from now on. G will be a compact Hausdorff topological group, X a locally compact Hausdorff space (often compact) with Radon measure μ. We assume (G, X) is a left-transformation group [3] such that G acts freely. Define π: $X \to X/G \equiv Y$ to be the canonical projection, and let $\nu = \pi(\mu)$. If $A \subset X$, let $G \cdot A = \{g \cdot x | g \in G, x \in A\}$. If $f \in C(X)$, let $(f \cdot g)(x) = f(gx)$; if $\mu \in C^*(X)$, let $(g\mu)(f) = \mu(fg)$. We will sometimes write dg for Haar measure on G. Finally, in §§4–6, T will denote an *arbitrary* group such that (X, T) is a *right* transformation group; in §§5–6, (G, X, T) will be a *bitransformation* group (the actions of G and T commute). See [3].

I. COMPACT TRANSFORMATION GROUPS

In §1, we assume G is a Lie group and X is compact. In §2, G is allowed to be metric; in §3, X becomes locally compact.

1. **X compact, G Lie.** Let μ be a measure on X, $\nu = \pi(\mu)$. To disintegrate μ with respect to π, we first express X as a "measurable product" $Y \times G$. We then apply the Dunford-Pettis theorem (0.7) to the map ζ: $L^1(\nu) \to C^*(G)$ given by $\zeta(f) = \pi_2[(f \circ \pi) \cdot \mu]$; here π_2: $X \to G$ is the projection. The map ω: $Y \to C^*(G)$ that results will then be used to construct a disintegration of μ.

The key to this section is the following result; it is an immediate corollary of [8, §5.4, Theorem 1].

1.1. Theorem. *For each $x \in X$, there is a compact neighborhood U of x and a compact $F \subset U$ such that $\pi^{-1}(y) \cap F$ is a single point whenever $y \in \pi(U)$.*

1.2. *Notation*. For each $x \in X$, pick sets U_x, F_x satisfying the conditions of 1.1. It is clear that we can replace each U_x by its saturation $G \cdot U_x$. Assume this done, and choose sets U_{x_i} $(1 \leq i \leq r)$ which cover X. Let $U_i \equiv U_{x_i}$, $F_i \equiv F_{x_i}$, $V_i = \pi(U_i)$. Define maps τ_i: $V_i \to U_i$ by $\{\tau_i(y)\} = F_i \cap \pi^{-1}(y)$. Let $A_1 = V_1, A_i = V_i \sim \bigcup_{j=1}^{i-1} V_j$ $(2 \leq i \leq r)$. Then the A_i are Borel, and $\bigcup_{i=1}^{r} A_i = Y$. Let $B_i = \pi^{-1}(A_i)$. Define τ: $Y \to X$ by $\tau|_{A_i} = \tau_i$.

The following lemma is a consequence of the definitions.

1.3. Lemma. *The maps* $(g, x) \to g \cdot x$: $G \times F_i \to U_i$ *and* v_i: $(g, y) \to g \cdot \tau_i(y)$: $G \times V_i \to U_i$ *are homeomorphisms* $(1 \leq i \leq r)$. *The map* τ *is a section of* X *over* Y *(i.e.,* $\tau(y)$ *is an element of* $\pi^{-1}(y)$ *for each* $y \in Y$*), and* τ *is* ν*-Lusin-measurable.*

1.4. Definition. For $x \in X$, let $\pi_2(x) \in G$ be determined by $\pi_2(x) \cdot \tau(y) = x$ $(y = \pi(x))$; thus π_2: $X \to G$.

In other words, $\pi_2(X)$ is that g such that $g \cdot \tau(y) = x$. Using 1.3, one sees that π_2 is continuous on each set B_i. By 0.2, π_2 is η-Lusin measurable for every $\eta \in C^*(X)$. Also, if $K \subset G$ is compact, then $\pi_2^{-1}(K)$ is η-integrable $(\eta \in C^*(G))$. Therefore,

1.5. Lemma. *The map* π_2 *is* η*-proper* (0.4) *for every* $\eta \in C^*(G)$.

1.6. Theorem. *There is a* ν*-Lusin-measurable map* ω: $Y \to M_+(G)$: $y \to \omega_y$ *such that*:

(a) $\|\omega_y\| = 1$ $(y \in Y)$.

(b) *If* $f \in M^1(Y, \nu)$ *and* $h \in C(G)$, *then*

$$\int_X f(\pi(x))h(\pi_2(x))\, d\mu(x) = \int_Y f(y)\omega_y(h)\, d\nu(y).$$

If ω': $Y \to C^*(G)$: $y \to \omega_y'$ *is another* ω^**-*ν*-measurable map satisfying* (b) *such that* $\|\omega_y'\| \leq M$ ν*-a.e. for some* M, *then* $\omega_y' = \omega_y$ ν*-a.e.*

Proof. Let $f \in M^1(Y, \nu)$. Since π_2 is $(f \circ \pi) \cdot \mu$-proper (0.4), $\pi_2[(f \circ \pi) \cdot \mu] \equiv \zeta(f)$ is a measure on G. We estimate the norm of $\zeta(f)$:

$$\begin{aligned}\|\zeta(f)\| &= \sup_{\|h\| \leq 1} |\zeta(f) \cdot h| \\ &= \sup_{\|h\| \leq 1} \left|\int_X f(\pi(x))h(\pi_2(x))\, d\mu(x)\right| \\ &\leq \int_X |f \circ \pi_2(x)|\, d\mu(x) = N_1(f).\end{aligned}$$

It follows immediately that ζ induces a linear map of $L^1(Y, \nu)$ into $C^*(G)$ such that $\|\zeta(f)\| \leq N_1(f)$. By the Dunford-Pettis theorem (0.7), there is a ν-Lusin-measurable map ω: $Y \to C^*(G)$: $y \to \omega_y$ such that (b) holds; further, if ω' satisfies the description in 1.6, then $\omega' = \omega$ ν-a.e.

It must still be shown that (perhaps after modification on a set of measure zero) one has $\omega_y \in M_+(X)$ and $\|\omega_y\| = 1$ for all y. Let $h \in C_+(G) = \{f \in C(G) | f(g) \geqslant 0 \text{ for all } g\}$, and let $Y \subset B_h = \{y | \omega_y(h) < 0\}$. Then B_h is ν-measurable (since ω is); let θ be the characteristic function of B_h. Then

$$0 \leqslant \int_X \theta(\pi(x))h(\pi_2(x))\, d\mu(x) = \int_Y \theta(y)\omega_y(h)\, d\nu(y) \leqslant 0.$$

Hence $\nu(B_h) = 0$. Let $(h_i)_{i=1}^{\infty}$ be a countable dense subset of $C_+(G)$; it is easily seen that $h \in C_+(G) \Rightarrow B_h \subset \bigcup_{i=1}^{\infty} B_{h_i}$. It follows that $\omega_y \geqslant 0$ ν-a.e. To check that $\|\omega_y\| = 1$ ν-a.e., let $h_0(g) \equiv 1$ $(g \in G)$. Then $\|\omega_y\| = \omega_y(h_0)$, but by (b),

$$\begin{aligned} \int_Y f(y)\, d\nu(y) &= \int_X f(\pi(x))\, d\mu(x) \\ &= \int_X f(\pi(x))h_0(\pi_2(x))\, d\mu(x) \\ &= \int_Y f(y)\omega_y(h_0)\, d\nu(y) \end{aligned}$$

for all $f \in M^1(\nu)$. It follows that $\omega_y(h_0) = 1$ ν-a.e.

We are still using the notation of 1.2.

1.7. Lemma. *Let $\varepsilon > 0$ and $f \in C(X)$ be given. For each i, $1 \leqslant i \leqslant r$, there exist $h_i \in C(G)$ and a bounded Borel function $\psi_i \colon Y \to C$ such that, on B_i, $|f(x) - \psi_i(\pi(x))h_i(\pi_2(x))| < \varepsilon$.*

Proof. Let $v_i \colon G \times V_i \to U_i$ be the homeomorphism of 1.3, and let $f_i = f \circ v_i$. There are functions $\psi_i' \in C(V_i)$, $h_i \in C(G)$ such that $|f_i(g, y) - \psi_i'(y)h_i(g)| < \varepsilon$ $(y \in V_i, g \in G)$. Let

$$\psi_i(y) = \begin{cases} \psi_i'(y), & y \in A_i, \\ 0, & y \notin A_i. \end{cases}$$

Then ψ_i is bounded Borel. Now on B_i, $\pi_2(x) = g$ and $\pi(x) = y$, where $v_i(g, y) = x$. Thus, on B_i, $|f(x) - \psi_i(\pi(x))h_i(\pi_2(x))| < \varepsilon$.

1.8. Definition. If $y \in Y$, let $\phi y \colon G \to X \colon g \to g \cdot \tau(y)$ (τ is defined in 1.3). Observe that ϕy is a homeomorphism onto $\pi^{-1}(y)$.

1.9. Theorem. *There exists a ν-Lusin-measurable disintegration $\lambda \colon Y \to M_+(X) \colon y \to \lambda_y$ of μ with respect to π. If $\lambda' \colon y \to \lambda_y'$ is another ν-Lusin-measurable map satisfying* 0.6 (c), (d) *and such that $\|\lambda_y'\| \leqslant M < \infty$ ν-a.e., then $\lambda_y' = \lambda_y$ ν-a.e.*

Proof. Let ω be the map of 1.6, and define a measure λ_y on X by $\langle \lambda_y, f \rangle = \langle \omega_y, f \circ \phi y \rangle$ $(f \in C(X))$. It follows immediately from 1.6 and this definition of λ_y that $\lambda_y \geqslant 0$, $\|\lambda_y\| = 1$, and $\mathrm{Supp}(\lambda_y) \subset \pi^{-1}(y)$. We must show that $\lambda: y \to \lambda_y$ is ν-Lusin-measurable, and that $\mu(f) = \int_Y \lambda_y(f)\, d\nu(y)$ for all $f \in C(X)$.

For measurability, fix $\varepsilon > 0$, and choose a compact $F \subset Y$ such that (i) $\nu(\sim F) < \varepsilon$, and (ii) both $\omega|_F$ and $\tau|_F$ are continuous. Let $f \in C(X)$, and suppose $y_n \to y$ in F. Then

$$\begin{aligned}|\lambda_{y_n}(f) - \lambda_y(f)| &= |\lambda_{y_n}(f \circ \phi y_n) - \lambda_y(f \circ \phi y)| \\ &\leqslant |\lambda_{y_n}(f \circ \phi y_n) - \lambda_{y_n}(f \circ \phi y)| + |\lambda_{y_n}(f \circ \phi y) - \lambda_y(f \circ \phi y)| \\ &\leqslant \|f \circ \phi y_n - f \circ \phi y\|_{C(G)} + |\lambda_{y_n}(f \circ \phi y) - \lambda_y(f \circ \phi y)|.\end{aligned}$$

By (ii) above, both terms tend to zero. Thus $\lambda_{y_n} \to \lambda_y$ (vaguely) $\Rightarrow \lambda|_F$ is continuous $\Rightarrow \lambda$ is ν-Lusin-measurable.

By ν-measurability and the fact that $\|\lambda_y\| = 1$ for all y, the formula

$$\mu'(f) = \int_Y \lambda_y(f)\, d\nu(y) \qquad (f \in C(X))$$

defines a measure on X (0.5). We will show that $\mu' = \mu$.

Observe first that

$$(*) \qquad \langle \lambda_y, h \circ \pi_2 \rangle = \langle \omega_y, h \rangle \qquad (y \in Y, h \in C(G)).$$

Let θ_i be the characteristic function of $A_i \subset Y$. Recalling that $B_i = \pi^{-1}(A_i)$, we have

$$\mu(f) = \sum_{i=1}^{r} \int_{B_i} f(x)\, d\mu(x) = \sum_{i=1}^{r} \int_X \theta_i(\pi(x)) f(x)\, d\mu(x) \qquad (f \in C(X)).$$

Fix f, and let $\varepsilon > 0$ be given. By 1.7, there are functions $h_i \in C(G)$ and bounded Borel functions ψ_i on Y such that $|f(x) - \psi_i(\pi(x)) h_i(\pi_2(x))| < \varepsilon$ $(x \in B_i)$. Let f' be defined by

$$f'(x) = \psi_i(\pi(x)) h_i(\pi_2(x)) \qquad (x \in B_i, 1 \leqslant i \leqslant r).$$

Then f' is μ'-integrable, so

$$\begin{aligned}\mu'(f') &= \int_Y \lambda_y(f')\, d\nu(y) \quad \text{(by 0.5)} \\ &= \sum_{i=1}^{r} \int_Y \theta_i(y)\psi_i(y)\lambda_y(h \circ \pi_2)\, d\nu(y) \\ &= \sum_{i=1}^{r} \int_Y \theta_i(y)\psi_i(y)\omega_y(h)\, d\nu(y) \quad \text{(by $(*)$ above)} \\ &= \sum_{i=1}^{r} \int_Y \theta_i(\pi(x))\psi_i(\pi(x))h(\pi_2(x))\, d\mu(x) \quad \text{(by 1.6)} \\ &= \mu(f').\end{aligned}$$

It now follows easily from the uniform bound $|f(x) - f'(x)| < \varepsilon$ $(x \in X)$ that $\mu(f) = \mu'(f) = \int_Y \lambda_y(f)\, d\nu(y)$.

Uniqueness remains to be shown. Let λ' be as in the statement of 1.8, and let $\omega'_y = \pi_2(\lambda'_y)$. It is straightforward to check that $y \to \omega'_y$ is ν-Lusin-measurable. Let $f \in M^1(\nu)$, $h \in C(G)$; then $(f \circ \pi) \cdot (h \circ \pi_2)$ is μ-integrable, hence (0.5)

$$\begin{aligned}((f \circ \pi) \cdot (h \circ \pi_2)) &= \int_Y \lambda'_y((f \circ \pi) \cdot (h \circ \pi_2))\, d\nu(y) \\ &= \int_Y f(y) \lambda'_y(h \circ \pi_2)\, d\nu(y) \\ &= \int_Y f(y) \omega'_y(h)\, d\nu(y).\end{aligned}$$

By uniqueness in 1.6, $\omega'_y = \omega_y$ ν-a.e., and it follows that $\lambda'_y = \lambda_y$ ν-a.e.

2. X **compact**, G **metric**. The following result ([8, Chapter IV, §7] or [5, p. 67]) is basic.

2.1. THEOREM. *Let H be a compact topological group. Then every neighborhood of the identity element contains a closed normal subgroup L such that H/L is a Lie group.*

2.2. *Notation*. Consider a transformation group (G, X) where G is not necessarily metric. Let $\{G_l\}$ be a decreasing net of closed normal subgroups such that G/G_l is a Lie group; such exists by 2.1. If G *is* metric, $\{G_l\}$ may be taken to be a sequence. Let $X_l = X/G_l$, π_l: $X \to X_l$; observe that $(G/G_l, X_l)$ is a transformation group with G/G_l Lie. Each space $C(X_l)$ may be embedded in $C(X)$; one then has that $\bigcup_l C(X_l)$ is dense in $C(X)$.

2.3. *Discussion*. Let (G, X) be a transformation group with G metric, $\{G_l\}$ a decreasing sequence of normal subgroups as in 2.2. Let μ be a measure on X, $\mu_l = \pi_l(\mu)$. Apply 1.9 to a fixed μ_l to obtain a ν-Lusin-measurable disintegration $\tilde{\lambda}^l$: $Y \to M_+(X_l)$: $y \to \tilde{\lambda}^l_y$. Define a map λ^l: $Y \to M_+(X)$ by *Haar-lifting* elements $\tilde{\lambda}^l_y$ by G_l; thus

$$\lambda^l_y(f) = \int_{X_l}\left(\int_{G_l} f(g_l x)\, dg_l\right) d\tilde{\lambda}^l_y,$$

where dg_l refers to Haar measure on G_l (observe that the quantity in parentheses defines a continuous function on X_l when $f \in C(X)$). It is easily checked that λ^l is ν-Lusin-measurable.

2.4. THEOREM. *There is a ν-Lusin-measurable disintegration λ of μ with respect to π. The uniqueness statement of* 1.9 *holds here, also.*

PROOF. Let l, m be positive integers, $l > m$, and let $A_{l,m} = \{y \in Y | \lambda^l_y(f)$

$= \lambda_y^m(f)$ for all $f \in C(X_m)\}$. Since $\lambda^l|_{C(X_m)}$ is clearly a ν-Lusin-measurable disintegration of μ_m, uniqueness in 1.9 implies that $\nu(A_{l,m}) = 1$. Let $A = \bigcap_{l>m} A_{l,m}$; A also has ν-measure 1. Now if $f \in Q = \bigcup_{l=1}^{\infty} C(X_l) \subset C(X)$, and if $y \in A$, then there is an l_0 so that $l \geqslant l_0 \Rightarrow \langle f, \lambda_y^l \rangle$ is constant. That is, $\lim_{l\to\infty} \langle f, \lambda_y^l \rangle$ exists for each $f \in Q$. Since Q is dense in $C(X)$, we conclude that $\lambda_y \equiv \lim_{l\to\infty} \lambda_y^l \in M_+(X)$ exists for each $y \in A$. Define $\lambda\colon Y \to M_+(X)$ to be λ_y if $y \in A$, and some point-mass supported on $\pi^{-1}(y)$ if $y \notin A$.

We show that λ is ν-Lusin-measurable (this is not immediate, since $M_+(X)$ is not separable). Let $0 < \varepsilon < 1$, and choose compact sets $K_l \subset A \subset Y$, $\nu(K_l) > 1 - \varepsilon \cdot 2^{-(l+1)}$, such that $\lambda^l|_{K_l}$ is continuous. Let $K = \bigcap_{l=1}^{\infty} K_l$; then $\nu(K) > 1 - \varepsilon$. If $y_n \to y$ in K, and if $f \in C(X_l)$, then

$$\lambda_{y_n}(f) = \lambda_{y_n}^l(f) \to \lambda_y^l(f) = \lambda_y(f).$$

It follows that λ is ν-Lusin-measurable.

That $\mathrm{Supp}(\lambda_y) \subset \pi^{-1}(y)$ is a consequence of $\mathrm{Supp}(\lambda_y^l) \subset \pi^{-1}(y)$ $(l \geqslant 1)$. The other conditions of 0.6 obviously hold. The uniqueness is obtained as follows. Let σ be another ν-Lusin-measurable disintegration of μ, and let $\sigma_l = \sigma|_{C(X_l)}$. Then σ_l defines a disintegration of μ_l, hence equals $\lambda|_{C(X_l)}$ on a set B_l of ν-measure 1. Let $B = \bigcap_{l=1}^{\infty} B_l$; $\sigma = \lambda$ on B.

3. *X* **locally compact,** *G* **metric.** For the material in 3.1–3.5, see [1, Chapter IV, §5, N°s 9, 10].

3.1. Definition. A family $(A_i)_{i\in I}$ of subsets of Y is *locally countable* if, for each $y \in Y$, there is a neighborhood V of y such that $V \cap A_i$ is nonempty for at most countably many i.

3.2. Theorem. *There is a locally countable family $(K_i)_{i\in I}$ of compact subsets of Y, pairwise disjoint, such that $Y \sim \bigcup_{i\in I} K_i$ is locally ν-null.*

3.3. Definition. Let $A \subset Y$ be ν-measurable, and let f map A to a topological space Z. Say that f is *ν-measurable* if every extension of f to Y, constant on $Y \sim A$, is ν-Lusin-measurable.

3.4. Theorem. *Let $(A_i)_{i\in I}$ be a locally countable family of ν-measurable subsets of Y, $A = \bigcup_{i\in I} A_i$. A map $f\colon A \to Z$ is ν-measurable iff $f|A_i$ is measurable for each i.*

3.5. Theorem. *Let $K \subset Y$ be compact, $f\colon K \to Z$ a map. Then f is ν-measurable iff f is $\nu|_K$-Lusin-measurable. Here $\nu|_K$ is the restriction of ν to K.*

Choose a locally countable collection (K_i) of disjoint compact subsets of Y as in 4.2, and let $L_i = \pi^{-1}(K_i)$ for each i. Note $L_i \subset X$ is compact. Let $\tau_i\colon L_i \to X$ be the injection for fixed i; the induced map $\tau_i\colon M_+(L_i) \to M_+(X)$ is vaguely continuous. Define a map λ^i as follows: if $\tilde{\lambda}^i\colon K_i \to M_+(L_i)$ is a disintegration of $\mu|_{L_i}$ (as in 2.4) with respect to $\pi_i\colon L_i \to K_i$, let $\lambda^i = \tau_i \circ \tilde{\lambda}^i$.

Then λ^i is $\nu|_{K_i}$-measurable. Define λ: $Y \to M_+(X)$ by: $\lambda_y = \lambda_y^i$ if $y \in K_i$; $\lambda_y = \delta_x$ for some $x \in \pi^{-1}(y)$ if $y \notin \bigcup_{i \in I} K_i$.

3.6. THEOREM. *The map λ is a ν-Lusin-measurable disintegration of μ with respect to π. The uniqueness statement of* 1.9 *holds with "ν-a.e." replaced by "locally ν-a.e.".*

PROOF. Let $N = Y \sim \bigcup_{i \in I} K_i$. *Any* function defined on N is ν-measurable since N is locally-ν-null. Combining 3.5, 3.4, and 3.3 shows that λ is ν-Lusin-measurable. Clearly λ_y is supported on $\pi^{-1}(y)$, and $\|\lambda_y\| = 1$ $(y \in Y)$.

Let $f \in K(X)$ be nonnegative. Observe that, since $S = \text{Support}(f)$ is compact, the function $r(y) = \lambda_y(f)$ is ν-integrable. Also, the set $J = \{i \in I: S \cap L_i \neq \varnothing\}$ is countable. So:

$$\int_Y \lambda_y(f)\, d\nu(y) = \int_Y \sum_{j \in J} r(y) \cdot \phi_{K_j}\, d\nu(y) = \sum_{j \in J} \int_{K_j} r(y)\, d(\nu|_{K_j})(y)$$

$$= \sum_{j \in J} \int_{K_j} \tilde{\lambda}_y^j (f|_{K_j})\, d(\nu|_{K_j})(y) = \sum_{j \in J} \int_{L_j} (f \circ \tau_j)\, d(\mu|_{L_j}) = \int_X f\, d\mu.$$

This shows that μ is a ν-Lusin-measurable disintegration of μ with respect to π.

To show uniqueness, let λ' be another ν-Lusin-measurable disintegration. Restricting λ' to K_i for each i and applying uniqueness in 2.4 shows that $\lambda' = \lambda$ locally ν-a.e.

II. ERGODIC MEASURES ON BITRANSFORMATION GROUPS

4. Generalities on ergodic measures. We give some basic material, then prove a lemma (4.4) which is of importance in §5.

Let (X, T) be a transformation group with X compact Hausdorff and T an *arbitrary* group. If $t \in T$ and $A \subset X$, define $A \cdot t = \{xt | x \in X\}$. If $f \in C(X)$ and $t \in T$, let $(ft)(x) = f(xt)$ $(x \in X)$; if $\mu \in C^*(X)$, let $(t\mu)(f) = \mu(ft)$ $(f \in C(X))$.

4.1. DEFINITION. A measure μ on X is *T-invariant* if $t\mu = \mu$ for all $t \in T$. Then (0.5) $\mu(A \cdot t^{-1}) = \mu(A)$ for each μ-measurable $A \subset X$ and each $t \in T$.

4.2. DEFINITION. A measure μ on X is *T-ergodic* if: (i) it is a positive probability measure (i.e., $\|\mu\| = 1$); (ii) whenever $A \subset X$ is μ-measurable and $\mu(A \triangle At^{-1}) = 0$ for all $t \in T$, one has $\mu(A) = 0$ or $\mu(A) = 1$. Here $\triangle$ = symmetric difference.

It is easily seen that this definition is equivalent to the one obtained by replacing "A is μ-measurable" by "A is Borel".

We will later (§6) use the following well-known result; see Phelps [11] for a proof.

4.3. THEOREM. *A measure μ on X is ergodic iff μ is extreme in the compact convex set of T-invariant probabilities on X.*

4.4. LEMMA. *A measure μ on X is ergodic $\Leftrightarrow$ the following holds: if Z is a Hausdorff space and $f\colon X \to Z$ a μ-measurable map satisfying $f(xt) = f(x)$ μ-a.e. for each $t \in T$, then $f =$ constant μ-a.e.*

PROOF. $\Leftarrow$: Let $A \subset X$ be a μ-measurable with $\mu(A \triangle At^{-1}) = 0$ $(t \in T)$. If ϕ_A is the characteristic function of A, then $\phi_A(xt) = \phi_A(x)$ μ-a.e. for each t. By the hypothesis, one now obtains $\phi_A(x) = 0$ or 1 for μ-almost all x.

$\Rightarrow$: Let $Q_1 = \{z \in Z \mid$ there exists an open set V containing z such that $\mu(f^{-1}(V)) = 0\}$, and let $Q = \sim Q_1$. There are three steps: (i) Q contains at most one point; (ii) Q contains at least one point; (iii) if $Q = \{b\}$, then $f(x) = b$ μ-a.e.

(i) If $a, b \in Q$, $a \neq b$, choose disjoint open sets V_a, V_b containing a, b respectively, and let $A = f^{-1}(V_a)$, $B = f^{-1}(V_b)$. Then $A \cap B = \varnothing$, and $\mu(A) > 0$, $\mu(B) > 0$. Since $\mu(X) = 1$, one has $0 < \mu(A) < 1$, $0 < \mu(B) < 1$. However, $f(xt) = f(x)$ μ-a.e. $(t \in T) \Rightarrow \mu(A \triangle At^{-1}) = 0 = \mu(B \triangle Bt^{-1}) = 0$. Therefore ergodicity of μ is violated.

(ii) Suppose $Q = \varnothing$. Let K be a compact subset of X, $\mu(K) > 0$, such that $f|_K$ is continuous. Then $K_1 = f(K)$ is compact, Each $z \in K_1$ has a neighborhood V_z in Z such that $\mu(f^{-1}(V_z)) = 0$. Let $K_1 \subset \bigcup_{i=1}^n V_{z_i}$; then

$$0 < \mu(K) \leqslant \mu\big(f^{-1}(K_1)\big) \leqslant \mu\left[f^{-1}\left(\bigcup_{i=1}^{n} V_{z_i}\right)\right]$$
$$\leqslant \sum_{i=1}^{n} \mu\big(f^{-1}(V_{z_i})\big) = 0.$$

Thus $Q \neq \varnothing$, and there is a $b \in Z$ such that $Q = \{b\}$.

(iii) We show that, given $\varepsilon > 0$, there is a set K, $\mu(K) > 1 - \varepsilon$, such that $f(K) = \{b\}$. Let L be a compact set such that $\mu(L) > 1 - \varepsilon$ and $f|_L$ is continuous. Let μ_L be the restriction of μ to L, and let the support of μ_L be $K \subset L$. If μ_K is μ restricted to K, then Supp $\mu_K = K$, and if V is open in K, then $0 < \mu_K(V) = \mu(V)$ [1]. Further, $\mu(K) = \mu(L) > 1 - \varepsilon$ [1]. We claim that $f(K) = \{b\}$. For let $c \in f(K)$, $c \neq b$. Let V_c be a Z-open set containing b' such that $\mu(f^{-1}(V_c)) = 0$. Then $(f|_K)^{-1}(V_c) = K \cap f^{-1}(V_c)$ is open in K and has μ-measure zero. This contradiction establishes that $f(K) = \{b\}$.

5. **The disintegration of an ergodic measure.** Let (G, X, T) be a bitransformation group with G compact metric, and let μ be a T-ergodic measure on X. Let $\nu = \pi(\mu)$, and let $G_0 = \{g \in G \mid g\mu = \mu\}$. Then G_0 is a closed subgroup of G. By 4.3, μ has a ν-Lusin-measurable disintegration λ with respect to π. If

$x \in X$, let $\phi_x: G \to X: g \to g \cdot x$ be the associated homeomorphism onto $\pi^{-1}\pi(x)$.

5.1. DEFINITION. Let $H: X \to C^*(G)$ be given by $\langle H(x), \hat{h} \rangle = \langle \lambda_y, h \circ \phi_x^{-1} \rangle$ where $\hat{h}(g) = h(g^{-1})$ ($g \in G$, $x \in X$, $y = \pi(x)$, $h \in C(G)$).

Here $h \circ \phi_x^{-1}$ is assumed to be continuously extended to all of X; the choice of the extension does not matter because λ_y is supported on $\pi^{-1}(y)$.

5.2. PROPOSITION. (a) $H(gx) = g \cdot H(x)$ ($g \in G$, $x \in X$).

(b) $H(xt) = H(x)$ *μ-a.e.* ($t \in T$).

(c) *H is μ-Lusin-measurable.*

PROOF. (a) This follows immediately from the definition.

(b) Fix $t \in T$, and define $\omega: Y \to C^*(X): \omega_y = t^{-1}(\lambda_{yt})$. Since ν is T-invariant, it is easily seen that ω is ν-Lusin measurable; by 0.5, the formula $\eta(f) = \int_Y \omega_y(f)\, d\nu(y)$ ($f \in C(X)$) defines a measure η on X. Now

$$\eta(f) = \int_Y \langle \lambda_{yt}, ft^{-1} \rangle \, d\nu(y) = \int_Y \langle \lambda_y, ft^{-1} \rangle \cdot t d\nu(y)$$

$$= \int_Y \langle \lambda_y, ft^{-1} \rangle \, d\nu(y) \quad \text{(by 0.4)}$$

$$= \mu(f).$$

By uniqueness in 3.3, $\omega = \lambda$ ν-a.e. One now checks that $\langle H(xt), \hat{h} \rangle = \langle \omega_y, h \circ \phi_x^{-1} \rangle$ ($h \in C(G)$); the conclusion follows.

(c) It suffices to prove (c) when G is a Lie group. For, let G_l, X_l, and λ^l ($l \geq 1$) be as in §2, and define H_l as in 5.1 by replacing (G, X, T) by $(G/G_l, X_l, T)$ and λ and λ^l. Since $\lambda_y = \lim_{l \to \infty} \lambda_y^l$, one has $H(x) = \lim_{l \to \infty} H_l(x)$ ($x \in X, y = \pi(x)$). Since $C^*(G)$ is separable, H is μ-Lusin-measurable if the H_l are.

Assume G is Lie. By 1.3, there is a ν-measurable section $\tau: Y \to X$. Fix $0 < \varepsilon < 1$, and let $B \subset Y$ be a compact set such that $\nu(B) > 1 - \varepsilon$ and such that both $\tau|_B$ and $\lambda|_B$ are continuous. It is enough to show that H is continuous when restricted to $\pi^{-1}(B) = A$, since $\mu(A) = \nu(B) > 1 - \varepsilon$. So, let $x_n \to x_0$ in A, and let $h \in C(G)$. It must be verified that $\langle \lambda_{y_n}, \phi_{x_n}^{-1} h \rangle \to \langle \lambda_y, \phi_{x_0}^{-1} h \rangle$ if $y_n = \pi(x_n), y = \pi(x_0)$.

Observe that the map $\zeta: G \times B \to A: (g, y) \to g \cdot \tau(y)$ is continuous and bijective, hence a homeomorphism. Let $x_n = \zeta(g_n, y_n)$, $x_0 = \zeta(g_0, y_0)$. On A, define functions f_n, f by $f_n \circ \zeta(g, y) = h(gg_n^{-1})$, $f \circ \zeta(g, y) = h(gg_0^{-1})$. Then $f_n \to f$ uniformly on A. Extend f_n, f continuously to X, calling the extensions f_n, f also (f_n may not converge uniformly to f *on* X, but this will not matter).

It may be checked that $\phi_{x_n}^{-1} h = f_n | \pi^{-1}(y_n)$, $\phi_x^{-1} h = f | \pi^{-1}(y)$. Thus the proof will be completed if it is shown that $\langle \lambda_n, f_n \rangle \to \langle \lambda_y, f \rangle$. But

$$|\langle \lambda_{y_n}, f_n \rangle - \langle \lambda_y, f \rangle| \leq |\langle \lambda_{y_n}, f_n \rangle - \langle \lambda_{y_n}, f \rangle| + |\langle \lambda_{y_n}, f \rangle - \langle \lambda_y, f \rangle|.$$

Since $\|\lambda_{y_n}\| = 1$ and $f_n \to f$ uniformly on $\pi^{-1}(y_{n'})$, the first term tends to zero. The second term goes to zero because $\lambda|_B$ is continuous.

From 4.4, 5.2(b), and 5.2(c), we see that $H(x) =$ const μ-a.e. To identify the constant we use the following result; it is a corollary of the proofs of 1.5.4 and 1.5.5(1) in [9]. Let $\xi(\nu) = \{\tau \in M_+(X) | \tau \text{ is } T\text{-ergodic}, \pi(\tau) = \nu\}$.

5.3. THEOREM. *The map $g \to g\mu$: $G \to \xi(\nu)$ induces a homeomorphism of G/G_0 (left coset space) onto $\xi(\nu)$. If $f \in C(X)$, then $\mu(f) = \int_{G/G_0}(g'\mu)(f)\, d\delta_{[G_0]}(g'G_0)$ gives an integral representation of μ over $\xi(\nu) \cong G/G_0$. Here $\delta_{[G_0]}$ is the Dirac measure at the coset $[G_0]$. This representation is unique: if $\mu(f) = \int G/G_0(g'\mu)(f)\, d\eta(g'G_0)$ $(f \in C(X))$, then $\eta = \delta_{[G_0]}$.*

5.4. PROPOSITION. *$H(x) = \gamma_0$ μ-a.e., where γ_0 is Haar measure on G_0.*

PROOF. We know that $F(x) = \beta$ μ-a.e. for some $\beta \in C^*(G)$. By mimicking the proof of 5.2(b), it may be seen that $F(g_0x) = F(x)$ $(g_0 \in G_0)$; combining this with 5.2(a), one has $g_0 \cdot \beta = \beta$. If we show that Supp $(\beta) \subset G_0$, it will follow that $\beta = \gamma_0$.

Define $\hat{\beta} \in M_+(G)$ by $\hat{\beta}(h) = \int_G h(g^{-1})\, d\beta(g)$. The measure $\hat{\beta}$ induces a measure $\tilde{\beta}$ on G/G_0. Fix $f \in C(X)$. On the set A_μ of μ-measure 1 where $H(x) = \beta$, one has $\langle \lambda_y, f\rangle = \langle \hat{\beta}, f \circ \phi_x\rangle$ (5.1). Therefore

$$\begin{aligned}
\mu(f) &= \int_Y \langle \lambda_y, f\rangle\, d\nu(y) \\
&= \int_X \langle \hat{\beta}, f \circ \phi_x\rangle\, d\mu(x) \\
&= \int_G \int_X f(gx)\, d\mu(x)\, d\hat{\beta}(g) = \int_G (g\mu)(f)\, d\hat{\beta}(g) \\
&= \int_{G/G_0} (g\mu)(f)\, d\tilde{\beta}(gG_0).
\end{aligned}$$

By uniqueness in 5.3, $\tilde{\beta} = \delta_{[G_0]}$; it follows that $\hat{\beta}$, and hence β, is supported on G_0.

5.5. REMARK. Proposition 5.4 gives a precise formulation of the fact that "each λ_y looks like γ_0"; in fact, for ν-a.a. $y \in Y$ and $x \in \pi^{-1}(y)$, one sees that $\phi_x^{-1}(\lambda_y) = \gamma_0 \cdot g_x$ for some $g_x \in G$. Here $(\gamma_0 \cdot \tilde{g})(h) = \int_G h(\tilde{g}g)\, d\gamma_0(g)$. If $H(x) = \gamma_0$, then $\phi_x^{-1}(\lambda_y) = \gamma_0$.

We indicate some corollaries of 5.4.

5.6. DEFINITION. An *ergodic decomposition* of $\xi(\nu)$ is a collection $\{A_\eta | \eta \in \xi(\nu)\}$ of pairwise disjoint Borel sets such that $\eta(A_\eta) = 1$ and $\eta(A_\eta \triangle A_\eta t^{-1}) = 0$ $(t \in T)$.

This relaxes the usual definition, according to which the A_η would be *strictly* T-invariant.

We will use the set $A_\mu = \{x|H(x) = \gamma_0\}$ to obtain an ergodic decomposition of $\xi(\nu)$ which "splits up" fibers $\pi^{-1}(y)$ in a nice way.

5.7. LEMMA. (a) *If* $x \in A_\mu$, *then* $gx \in A_\mu$ *iff* $g \in G_0$; *i.e.*, A_μ *is* G_0-*saturated. If* $y = \pi(x)$, *then* $A_\mu \cap \pi^{-1}(y)$ *is homeomorphic to* G_0.
(b) $\mu(A_\mu \triangle A_\mu t^{-1}) = 0$ $(t \in T)$.

PROOF. (a) Note $H(gx) = g\gamma_0$, which equals γ_0 iff $g \in G_0$. This implies that $gx \in A\mu$ iff $g \in G_0$, which in turn implies that $\phi_x(G_0) = A_\mu \cap \pi^{-1}(y)$.
(b) This follows from 5.2(b).

5.8. REMARK. There is a set $A'_\mu \subset A_\mu$, $\mu(A'_\mu) = 1$, such that A'_μ is an F_σ in X and 5.7 holds with A'_μ in place of A_μ. To see this, write $A_\mu = \bigcup_{i=1}^\infty K_i \cup N$ where K_i is compact $(i > 1)$ and $\mu(N) = 0$. Let $A'_\mu = G_0 \cdot \bigcup_{i=1}^\infty K_i = \bigcup_{i=1}^\infty G_0 \cdot K_i \subset A_\mu$ (since A_μ is G_0-saturated). Then each $G_0 \cdot K_i$ is compact, so A'_μ is an F_σ; it is easily shown that 5.7 remains valid with A'_μ replacing A_μ. Replace A_μ by A'_μ, retaining the notation A_μ.

5.9. PROPOSITION. $\{gA_\mu|g \in G\}$ *is an ergodic decomposition of* $\xi(\nu)$; *if* $\eta = g\mu$, *then* $A_\eta = g \cdot A_\mu$.

The notation is meant to indicate the class of *distinct* sets gA_μ.

PROOF. By 5.7(a), $g \cdot A_\mu = A_\mu$ if $g \in G_0$, $gA_u \cap A_\mu = \varnothing$ if $g \notin G_0$, i.e., if $g\mu \neq \mu$. Thus $\{gA_\mu|g \in G\}$ is a pairwise disjoint collection of Borel sets. Also $(g\mu)(gA_\mu) = \mu(g^{-1}gA_\mu) = 1$, and $(g\mu)(gA_\mu \triangle gA_\mu \cdot t^{-1}) = \mu(A_\mu \triangle A_\mu t^{-1}) = 0$ (5.7). By 5.3, $\xi(\nu_0)$ is exactly $\{g\mu|g \in G\}$. Thus all conditions of 5.6 are satisfied.

We state without proof a theorem which depends on 5.9. Let $\bar\mu$ be the *Haar lift* of ν; i.e., $\mu(f) = \int_Y(\int_G f(gx)\, d\gamma(g))\, d\nu(y)$ $(f \in C(X))$. Observe that $\bar\mu$ is G-invariant, i.e., $g\bar\mu = \bar\mu$ for all g. Hence there is defined a natural unitary representation $(G, L^2(X, \bar\mu))$ of G on $L^2(X, \mu)$ via the formula $(g \cdot f)(x) = f(gx)$. Similarly, there is a unitary representation $(G_0, L^2(X, \mu))$.

5.10. THEOREM. $(G, L^2(X, \bar\mu))$ *is the representation induced* (*see* [4]) *by* $(G_0, L^2(X, \mu))$.

The proof is contained in [7]. Using this result, one may define and discuss a generalization of "functions of type γ" [10].

5.11. *Question.* If $T =$ integers or reals, ergodic sets have an interpretation (indeed, may be defined) in terms of regular points. May the ergodic sets of 5.6 be interpreted in some analogous way?

6. G nonmetrizable, Y has strong lifting.

6.1. We retain the assumptions and notation of §5, except that G need not be metric. We suppose that Support$(\nu) = Y$ and that ρ is a *strong lifting* of $L^\infty(Y, \mu)$. Thus ρ is a map from $M^\infty(Y, \nu)$ to $M^\infty(Y, \nu)$ such that: (i) ρ is linear; (ii) $\rho(f) = f$ ν-a.e. for all f; (iii) $f_1 = f_2$ ν-a.e. $\Rightarrow \rho(f_1) = \rho(f_2)$; (iv) $f \geq 0 \Rightarrow \rho(f) \geq 0$; (v) $\rho(f_1 \cdot f_2) = \rho(f_1) \cdot \rho(f_2)$; (vi) $\rho(1) = 1$; (vii) $\rho(f) = f$ for each $f \in C(Y)$ (see [6] for a complete discussion). It is the last property which is crucial; a function ρ satisfying (i)–(vi) always exists on $M^\infty(Y, \nu)$ [6, Chapter IV, Theorem 3].

Our goal is 6.9, which is an analogue of 5.4 (see also 5.5). We note that if a strong lifting of $L^\infty(Y, \nu)$ exists, then every extension $\bar{\nu}$ of ν has a weakly ν-measurable (0.5) disintegration with respect to π; see [6].

6.2. THEOREM. *Let λ: $Y \to M_+(X)$ be weakly ν-measurable and satisfy $\|\lambda_y\| <$ const $< \infty$ ν-a.e. There is a map λ': $Y \to M_+(X)$, satisfying the conditions just stated, such that $\lambda' = \lambda$ weakly a.e. and $\rho(\lambda') = \lambda'$.*

The last condition means that, if $f \in C(Y)$, the functions $\langle \lambda'_y, f \rangle$ and $\rho(\langle \lambda'_y, f \rangle)$ are equal for *all* $y \in Y$. For details and a proof, see [6, Chapter VI, §4]. We observe here that the conditions $\|\lambda'_y\| <$ const ν-a.e. and $\rho(\lambda') = \lambda'$, together with [6, Chapter IX, Proposition 5], show that λ' is ν-adequate (0.6(a)). Moreover, if $\|\lambda_y\| = 1$ for all y, then $1 = \rho(1) = \rho\langle \lambda_y, 1 \rangle = \langle \lambda'_y, 1 \rangle \Rightarrow \|\lambda'_y\| = 1$ for all y.

Recall that $G_0 = \{ g \in G \mid g\mu = \mu \}$; let $\xi_0 = \{ \eta \in M_+(X)\colon \|\eta\| = 1, g\eta = \eta \text{ for all } g \in G_0 \}$.

6.3. LEMMA. *Let $Z = X/G_0$, σ: $X \to Z$, the projection. Then σ induces an affine isomorphism of ξ_0 onto $M_+(Z) \cap \{ \tau \in M_+(Z) \mid \|\tau\| = 1 \}$.*

PROOF. We need only show that the induced map is bijective. To show injectivity, suppose $\sigma(\eta_1) = \sigma(\eta_2)$. Then $\eta_1(f) = \eta_2(f)$ for all $f \in C(X)$ satisfying $f \cdot g_0 = f (g_0 \in G_0)$. Pick $f \in C(X)$, and let $\tilde{f}(x) = \int_{G_0} f(g_0 x)\, dg_0$. Then $\tilde{f} \cdot g_0 = \tilde{f}$; further

$$\eta_1(\tilde{f}) = \int_X \int_{G_0} f(g_0 x)\, dg_0\, d\eta_1 = \int_{G_0} \int_X f(g_0 x)\, d\eta_1\, dg_0 = \eta_1(f),$$

and similarly $\eta_2(\tilde{f}) = \eta_2(f)$. One concludes that $\eta_1(f) = \eta_2(f)$. For surjectivity, pick $\tau \in M_+(Z)$, and let η be its G_0-Haar lift: $\eta(f) = \int_Y (\int_{G_0} f(g_0 x)\, dg_0)\, d\tau(y)$. Clearly $g\eta = \eta$ $(g \in G_0)$ and $\sigma(\eta) = \tau$.

6.4. PROPOSITION. *η is extreme in $\xi_0 \Leftrightarrow \eta(f \cdot h) = \eta(f) \cdot \eta(h)$ for all $f, h \in C(Z) \subset C(X)$.*

PROOF. If η is extreme, 6.3 implies that $\sigma(\eta)$ is also. Hence $\sigma(\eta)$ is a Dirac measure placed at some $z \in Z$, so η is multiplicative on $C(Z)$. On the other hand, if η is multiplicative on $C(Z)$, so is $\sigma(\eta) \Rightarrow \sigma(\eta)$ is extreme $\Rightarrow$ (by 6.3 again) η is extreme in ξ_0.

Combining 6.3 and 6.4 shows that those measures on X ergodic with respect to G_0 are the G_0-Haar lifts of Dirac measures on Z.

6.5. COROLLARY. *Let* ρ: $M^{\infty}(Y, \nu) \to M^{\infty}(Y, \nu)$ *satisfy* (i)–(vi) *of* 6.1 (*thus* ρ *is a lifting of* $M^{\infty}(Y, \nu)$). *Let* λ: $Y \to M_+(X)$ *be weakly* ν*-measurable, and suppose* λ_y *is extreme in* ξ_0 *for all* y. *Then* $\rho(\lambda)_y$ *is also extreme in* ξ_0 *for all* y.

PROOF. Let $f, h \in C(Z), \bar{y} \in Y$. Then

$$\rho(y)_{\bar{y}} \cdot (f \cdot h) = [\rho\langle\lambda_y, f \cdot h\rangle](\bar{y}) = [\rho\langle\lambda_y, f\rangle \cdot \rho\langle\lambda_y, h\rangle](\bar{y})$$
$$= (\rho(\lambda)_{\bar{y}} \cdot f)(\rho(\lambda)_{\bar{y}}, h)$$

by 6.1(vi) and 6.4. Thus $\rho(\lambda)_{\bar{y}}$ is multiplicative on $C(Z)$ for each $\bar{y}$, hence extreme in ξ_0.

The next lemma is used in the proof of 6.9; it will also allow us to tie 5.4 and 6.7 together. Observe that if η is extreme in ξ_0, then η is supported on $\pi^{-1}(\bar{y})$ for some $\bar{y} \in Y$.

6.6. LEMMA. *Let* $\eta \in M_+(X)$. *Then* η *is extreme in* ξ_0 *iff* $\phi_x^{-1}(\eta) = \gamma_0 \cdot g$ *for some* $g \in G$ $(x \in \pi^{-1}(\bar{y}))$..

PROOF. Follows easily from the fact that $\{\gamma_0 \cdot g | g \in G\}$ is precisely the set of measures on G ergodic with respect to the left action (defined by the group multiplication) of G_0 on G.

6.7. In what follows, we will use the notation of §2. Thus (G_l) is a decreasing net of closed normal subgroups of G such that G/G_l is Lie, and λ^l: $Y \to M_+(X)$ is obtained as in 2.3 (metrizability of G is not necessary there). Let $G_{0,l}$ consist of those $g \in G$ such that g projects to a point $\tilde{g} \in G/G_l$ satisfying $\tilde{g} \cdot \mu_l = \mu_l$. It is not hard to show that (i) each $G_{0,l}$ is a closed subgroup of G; (ii) $\bigcap_l G_{0,l} = G_0$; (iii) if γ_l is Haar measure on $G_{0,l}$, then $\gamma_l \to \gamma_0$ (vague convergence) in $M_+(G)$ (for the arguments here, see [7, Appendix A]). By use of 5.4 and 5.5, it is easily seen that (for fixed l and ν-a.a. y) $\phi_x^{-1}(\lambda_y^l) = \gamma_l \cdot g_x$ for some $g_x \in G$ $(x \in \pi^{-1}(y))$. Lemma 6.6 applies equally well if ξ_0 is replaced by $\xi_l = \{\eta \in M_+(X) | \, \|\eta\| = 1, g \cdot \eta = \eta$ for all $g \in G_{0,l}\}$; hence λ_y^l is extreme in ξ_l for ν-a.a. y. Modifying on a set of measure zero if necessary, we assume this is so for *all* y.

6.8. Let ρ be a strong lifting on (Y, ν). Applying 6.2 to replace each λ^l by a new map (again called λ^l) which is equal to the old one weakly ν-a.e., is weakly ν-measurable, and which satisfies $\rho(\lambda^l) = \lambda^l$. By 6.5, λ_y^l is extreme in ξ_l for all y. Since ρ is strong, $\mathrm{Supp}(\lambda_y^l) \subset \pi^{-1}(y)$ for all y.

6.9. THEOREM. *There is a weakly* ν*-measurable disintegration* λ *of* μ *with respect to* π *such that* $\lambda_y \in \xi_0$ *for all* $y \in Y$.

PROOF. We show first that $\lim_l \lambda_y^l$ exists for all $y \in Y$, and defines a weakly measurable map $\lambda: y \to \lambda_y$. Let $f \in C(X)$. Since $\bigcup_l C(X_l)$ is dense in $C(X)$, there is a sequence (f_n) in this union which converges to f uniformly. Fix n; there is an l_n such that $l > l_n \Rightarrow f \in C(X_l)$. We claim that, if $l > l_n$, then

$\langle \lambda_y^{l_n}, f_n \rangle = \langle \lambda_y^l, f_n \rangle \equiv h_n(y)$ for all y. To see this, note uniqueness in 1.9 and the definitions of the λ^l imply that $\langle \lambda_y^l, f_n \rangle = h_n(y)$ ν-a.e. But then $\langle \lambda_y^l, f_n \rangle = \rho\langle \lambda_y^l, f_n \rangle = \rho(h_n)(y) = h_n(y)$ for *all* y, proving the assertion. Simple estimates now show that (λ^l) is Cauchy in l (f is arbitrary), and that, if $\lambda_y = \lim \lambda_y^l$, then

$$(*) \qquad \langle \lambda_y, f \rangle = \lim_{n\to\infty} h_n(y)$$

(the limit is actually *uniform*). Thus λ exists and is weakly measurable.

To see that $\lambda_y \in \xi_0$ for fixed y, let $x \in \pi^{-1}(y)$. By 6.7, $\phi_x^{-1}(\lambda_y^l) = \gamma_l \cdot g_l$ for some $g_l \in G$. Using (iii) of 6.7 and choosing a convergent subsequence, we obtain $\phi_x^{-1}(\lambda_y) = \gamma_0 \cdot g$. By 6.6, $\lambda_y \in \xi_0$.

Now replace λ by $\rho(\lambda)$. The remarks after 6.2 and the fact that ρ is strong show that 0.6(a), (b), and (c) hold. To obtain 0.6(d), use $(*)$, the uniformity of the convergence, and the fact that $\mu_{l_n}(f_n) = \mu(f_n)$ for all n. By 6.5, we still have $\lambda_y \in \xi_0$ for all y. The proof is completed.

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